HIGHER HOMOTOPY-COMMUTATIVITY

BY
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Introduction. In topology, there is a significant class of H-spaces which are homotopy-commutative, but not strictly commutative. As examples, one has the loop spaces of H-spaces, for which, in general, no strictly commutative form exists. However, these loop spaces are known, cf. [15], to satisfy far stronger homotopy-commutativity properties than the usual one. It is the purpose of this paper to introduce a new sequence of strong forms of homotopy-commutativity, called C_n-forms, which are possessed by these loop spaces, and to examine the geometric implications of a space possessing such a form.

The classical Hopf construction characterizes the H-spaces as being those spaces which can be the null-homotopic fiber of a fibration over their own suspension. Further, it follows from work of Stasheff, [11], or by a direct construction of the present author, [17], that among the homotopy-associative H-spaces, the homotopy-commutative ones are those for which the Hopf construction extends to a fibration over the James' two-fold reduced product space of their suspension. Now spaces which are fibered over n-fold reduced product spaces are important; they have been used, for example, in the computation of homotopy groups, cf. Toda, p. 174 of [16]. It will be the principal result of this paper that those topological monoids for which the Hopf construction extends to a fibration over the n-fold reduced product space of their suspension are exactly those which admit C_n-forms. (Although it is possible to define the concept of C_n-form in the category of Stasheff's A_{n+1}-spaces, the overwhelming bulk of notation required in that case causes the present paper to deal with only the strictly associative case.)

The morphisms in the category of spaces with C_n-forms will be introduced, and these morphisms will be used to determine when certain induced fiber spaces admit C_n-forms. This, in turn, leads to a characterization in terms of k-invariants of certain spaces which admit C_n-forms.

The organization of this paper is as follows. The first section gives general background material and definitions. The concept of C_n-form is introduced in an intuitive fashion, the basic results of the paper are discussed informally, and indications are given as to future research possibilities in this area. The second section presents the precise definitions of the C_n-forms, states the main theorems, and gives applications. The proofs are contained in the third section.

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1. The concept of H-space is one which has been the subject of much research in recent years. (Recall that a topological space is called an H-space if there is a continuous map $m: X\times X \to X$, written $m(x, y)=xy$, and an element e in X such that $ex=x=xe$ for every x in X.) The importance of H-spaces has come about both because H-spaces arise in many more varied contexts than their parents, the topological groups, and because many of the results of the theory of topological groups remain valid in the more general context of H-spaces.

In many situations, the notions of associativity and commutativity of the multiplication in an H-space are too strict. In the case of associativity, the weaker property of homotopy-associativity (i.e., $(xy)z$ is homotopic to $x(yz)$) occurs more frequently in practice. Similarly, homotopy-commutativity (xy homotopic to yx) replaces strict commutativity in many cases. As was mentioned in the Introduction, a well-known example of this phenomenon occurs in the space of loops on an H-space.

One of the principal methods used in the investigation of H-spaces has been the interplay between their algebraic properties, such as those discussed in the preceding paragraph, and certain geometric properties. The first results of this nature were in the development of the Hopf construction, due to H. Hopf [4], and later extended by Spanier-Whitehead [10] and Sugawara [13]. This construction characterizes a space as being an H-space if and only if it can appear as the fiber in a certain type of fibration. The Hopf construction will be reviewed in §3. In [14], Sugawara extended this method to characterize homotopy-associative H-spaces, and the author, in [17], has given the analogous characterization of H-spaces which are both homotopy-associative and homotopy-commutative. Perhaps the most spectacular results in this line are those of Stasheff's theory of A_n-spaces [14], in which he determines homotopy-associativity criteria for deciding which H-spaces admit classifying spaces.

In discussing the two permutations of the product of two elements of an H-space, X, there is no need to worry about associativity of the product. However, when one begins to consider the various permutations of products of more than two elements of X, the problem of nonassociativity rears its ugly head. To dismiss this problem once and for all, the forms of higher homotopy-commutativity considered herein will be defined only for associative H-spaces, called topological monoids. In dealing with loop spaces, this is no real restriction since there is a well-known associative model of the space of loops, given as follows: Let X be a based space with base point x_0. Then the Moore space of paths in X, PX, consists of all pairs (λ, r), where λ: $[0, r]\to X$ and $\lambda(0)=x_0$. The Moore loop space of X, ΩX, consists of all (λ, r) in PX for which $\lambda(r)=x_0$. The pair (λ, r) will be denoted by λ^r.

Sugawara [15] characterized the loop space of an H-space by strengthened homotopy-commutativity designed to insure the existence of H-structure in its classifying space. These conditions are described in Definition 7 of §2 of this paper. It is not known what the implications of Sugawara's conditions are in the context of the Hopf-type constructions.

Before discussing the C_n-forms of homotopy-commutativity, we introduce some notation and terminology. All spaces, unless otherwise specified, will be assumed to have base points. In the case of an H-space, the base point will be taken to be the identity element, and in the case of a CW-complex, the base point will be assumed to be a 0-cell. The suspension of a space X, denoted by SX, will be the product of I (the unit interval) with X, with the subspace $(\partial I \times X) \cup (I \times \{x_0\})$ identified to a point. The n-fold Cartesian product of X with itself will be denoted by X^n. We are now prepared to discuss the higher forms of homotopy-commutativity.

The basic idea is simple and intuitive. A homotopy-commutativity is simply a homotopy connecting xy with yx, as described above. Now if a monoid, X, is homotopy-commutative, then a chain connecting the various permutations of a product of three elements of X is given as follows:

$$xyz \sim xzy \sim zxy \sim zyx \sim yzx \sim yxz \sim xyz.$$

The pattern here is that two permutations are connected by a homotopy which commutes adjacent elements. Thus a map is defined $\partial I^2 \times X^3 \to X$. If this map extends to a map $I^2 \times X^3 \to X$, then X satisfies the next higher degree of homotopy-commutativity. Ordinary homotopy-commutativity will be called a C_2-form, and this new map, if it exists, will be called a C_3-form on X, where the subscripts refer to the number of elements of X being permuted. Now if such a C_3-form exists, then using it a map may be defined from $\partial I^3 \times X^4 \to X$. If this extends to a new map $I^3 \times X^4 \to X$, then X admits the next higher form of homotopy-commutativity, and this map is called a C_4-form, and so on. In general, if a space admits C_i-forms, $2 \leqq i \leqq n-1$, then a map is defined from $\partial I^{n-1} \times X^n \to X$, by using these forms, which connects all the permutations of a product of n elements of X. If this map extends to $I^{n-1} \times X^n \to X$, this extension will be called a C_n-form. It takes some care to divide up properly the boundary of I^n so that the permutations will be connected in the right way. Although the basic idea is simple, the notation is somewhat complicated. These concepts occupy Definition 1, 2, 4, and 5 of §2.

In this paper, we exploit the C_n-forms to obtain theorems of the Hopf construction type which characterize monoids which admit them, c.f. Theorem 14. Further, obstruction theoretic and dimensionality arguments are used to obtain some technical results about spaces admitting C_n-forms, which are then applied to give a characterization of C_n-spaces in terms of their k-invariants.

Several areas of investigation on C_n-spaces lie ahead. In a future paper, the author will describe the relationship between the C_n notion of homotopy-commutativity and the generalized higher Whitehead products of Porter, [9]. We also intend to develop the homology analogue of C_n-commutativity so that more effectively computable invariants of C_n-spaces may be obtained.

2. In this section the precise definition of C_n-forms will be given and the main theorems about them will be stated. The first step will be to determine how to arrange the various permutations of a product of n elements of a monoid around the boundary of an $(n-1)$-cell. This motivates the following definitions.

Definition 1. Let $\mathbf{n}$ denote the sequence $(1, 2, \ldots, n)$. Subsequences of $\mathbf{n}$ will be denoted by symbols A_l, B_m, etc., where the subscripts denote the number of elements in the subsequence. The function $i_A\colon A_l \to \mathbf{n}$ will be the inclusion, and A will denote the composition $\mathbf{l} \to A_l \to \mathbf{n}$. An ordered pair, (A_l, B_m), of disjoint subsequences of $\mathbf{n}$ is called a (l, m) *partition*, or shuffle, of $\mathbf{n}$ if $\operatorname{Im}(i_A) \cup \operatorname{Im}(i_B) = \mathbf{n}$. Analogously, an ordered k-tuple $(A_{l_1}, \ldots, A_{l_k})$ of disjoint subsequences of $\mathbf{n}$ is an $(l_1, \ldots, l_k)$ *partition* if their union is equal to $\mathbf{n}$. We shall utilize the following description of a cell-complex structure on the $(n-1)$-cell which is due to Milgram, [Definition 4.1 of 8], who observed its importance in connection with interated loop spaces.

Definition 2. Consider $\mathbf{n} = (1, 2, \ldots, n)$ as a point in R^n. The symmetric group S_n acts on R^n by permuting the coordinates. Let K_n be defined to be the convex hull of the orbit of $\mathbf{n}$ under this action.

From this definition, it is immediate that K_n is homeomorphic to an $(n-1)$-dimensional cell. The next two theorems are Lemmas 4.2 and 4.3 of [8].

Theorem 3. *The cells, K_n, satisfy the following properties.*

1. *The boundary of K_n, L_n, is the union of $(n-2)$-cells which are in one-to-one correspondence with the 2-fold partitions of n.*

2. *If (A_l, B_m) is such a partition, then the cell of L_n corresponding to it is the image of $K_l \times K_m$ by*

$$V(A_l, B_m)\colon K_l \times K_m \to L_n,$$

a linear homeomorphism into L_n. ($V(A_l, B_m)[K_l \times K_m]$ will be denoted by $K_l^A \times K_m^B$.)

3. *Two such cells of L_n, say $K_l^A \times K_m^B$ and $K_r^{A'} \times K_s^{B'}$, intersect if and only if there are partitions (C_p, D_q) of $\mathbf{l}$ and (C'_j, D'_k) of $\mathbf{s}$ such that $(p, q, m) = (r, j, k)$, and the diagram*

$$\begin{array}{ccccc}
\mathbf{p} \times \mathbf{q} \times \mathbf{m} & \longrightarrow & C_p \times D_q \times B_m & \longrightarrow & \mathbf{l} \times B_m \\
\| & & & & \downarrow \\
\mathbf{r} \times \mathbf{j} \times \mathbf{k} & & & & A_l \times B_m \\
\downarrow & & & & \downarrow \\
A'_r \times C'_j \times D'_k & \longrightarrow & A'_r \times \mathbf{s} \longrightarrow A'_r \times B'_S & \longrightarrow & \mathbf{n}
\end{array}$$

is commutative. In this case, their intersection is given by the commutativity of the following diagram.

$$
\begin{array}{ccc}
K_p \times K_q \times K_m & \xrightarrow{V(C_p,\, D_q)\times 1} & K_l \times K_m \\
\| & & \\
K_r \times K_j \times K_k & & \Big\downarrow V(A_l,\, B_m) \\
\Big\downarrow 1 \times V(C'_j,\, D'_k) & & \\
K_r \times K_S & \xrightarrow{V(A'_r,\, B'_S)} & K_n
\end{array}
$$

The cell complexes K_i admit more structure, as given in the following lemma. The maps guaranteed by this lemma may be thought of as degeneracy operators.

LEMMA 4. *There are maps* $s_j\colon K_{n+1} \to K_n$, $j=1,\ldots,n+1$, *such that*
(1) *if* $j=A(p)$, *then the diagram*

$$
\begin{array}{ccc}
K_r \times K_s & \xrightarrow{V(A_r,\, B_s)} & K_{n+1} \\
s_p \times 1 \Big\downarrow & & \Big\downarrow s_j \\
K_{r-1} \times K_s & \xrightarrow{V(A'_{r-1},\, B'_s)} & K_n
\end{array}
$$

is commutative, where (A'_{r-1}, B'_s) *is the partition of* $\mathbf{n}$ *given by*

$$
\begin{array}{ccc}
A' & \xrightarrow{i_{A'}} & \mathbf{n} \\
\Big\downarrow & & \Big\downarrow \\
A-\{p\} & & (\mathbf{n+1})-\{j\} \\
\Big\downarrow & & \Big\downarrow \\
A & \xrightarrow{i_A} & (\mathbf{n+1})
\end{array}
\qquad\qquad
\begin{array}{ccc}
B'_s & \xrightarrow{i_{B'}} & \mathbf{n} \\
\Big| & & \Big\downarrow \\
\Big| & & (\mathbf{n+1})-\{j\} \\
\Big\downarrow & & \Big\downarrow \\
B_s & \xrightarrow{i_B} & (\mathbf{n+1})
\end{array}
$$

where the unlabelled arrows are injections;
(2) *if* $j=B(q)$, *then the diagram*

$$
\begin{array}{ccc}
K_r \times K_s & \xrightarrow{V(A_r,\, B_s)} & K_{n+1} \\
1 \times s_q \Big\downarrow & & \Big\downarrow s_j \\
K_r \times K_{s-1} & \xrightarrow{V(A'_r,\, B'_{s-1})} & K_n
\end{array}
$$

is commutative, where (A'_r, B'_{s-1}) *is defined in similar fashion to* (A'_{r-1}, B'_s) *in* (1) *and*

(3) *if* $i<j$, *then*

$$\begin{array}{ccc} K_{n+1} & \xrightarrow{s_j} & K_n \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle s_i} \\ K_n & \xrightarrow{s_{j-1}} & K_{n-1} \end{array}$$

is commutative.

It is now possible, using this terminology, to make precise the intuitive definition of the higher homotopy-commutativity which was discussed above. The following is the basic definition.

DEFINITION 5. Let X be a topological monoid and $n \geqq 1$.Then a C_n*-form* $\{Q_i\}$, $i=1,\dots,n$, on X is a set of maps $Q_i\colon K_i \times X^i \to X$ such that

(1) $Q_1\colon K_1 \times X \to X$ is the identity map of X,

(2) $Q_i(V(A_r, B_s)[\rho, \sigma], x_1, \dots, x_i) = Q_r(\rho, x_{A(1)}, \dots, x_{A(r)}) \cdot Q_s(\sigma, x_{B(1)}, \dots, x_{B(s)})$, where $\rho \in K_r$, $\sigma \in K_s$, and $x_1, \dots, x_i \in X$, and

(3) if $x_j = e$, the identity of X, then

$$Q_i(\tau, x_1, \dots, x_i) = Q_{i-1}(s_j(\tau), x_1, \dots, \hat{x}_j, \dots, x_i).$$

A space, X, together with a C_n-form on X, is called a C_n*-space*. It may happen that a space admits maps $\{Q_i\}$ for all $i \geqq 1$, such that $\{Q_i\}$, $i=1,\dots,n$, is a C_n-form for every n. In this case X is called a C_∞*-space.*

REMARK 6. The fact that the definitions of the Q_i are consistent comes from the following observations.

(1) If two cells of the boundary of K_i intersect, the consistency of the definition of Q_i along their intersection is verified by part (3) of Theorem 3.

(2) The consistency of conditions (2) and (3) with each other is verified by parts (1) and (2) of Lemma 4.

(3) The consistency of condition (3) of the definition is given by conclusion (3) of Lemma 4.

EXAMPLES.

(1) Any commutative topological monoid, X, is a C_∞-space, by setting

$$Q_n(T, x_1, \dots, x_n) = x_1 \cdots x_n$$

for all T in K_n, $x_1, \dots, x_n$ in X.

(2) A C_2-form is simply a commuting homotopy, so the C_2-spaces are the homotopy-commutative monoids.

(3) If Y is an H-space, and $X = \Omega Y$, the associative loop space of Y, then X is a C_∞-space. Although this fact follows from Example 4, a direct proof will be given in §3.

Before giving Example 4, we introduce some terminology which will be used later to determine the invariants provided by the C_n-forms. The next definition is due to Sugawara [15].

DEFINITION 7. Let Y and Z be monoids. Then a map $f\colon Y \to Z$ is called *strongly homotopy-multiplicative* if there exist maps

$$M_n\colon Y \times (I \times Y)^n \to Z, \quad n = 0, 1, \dots,$$

such that

(1) $M_0 = f$ and

(2) $$\begin{aligned} M_n(y_0, t_1, y_1, \dots, t_n, y_n) &= M_{n-1}(y_0, t_1, y_1, \dots, t_{i-1}, y_{i-1}y_i, t_{i+1}, y_{i+1}, \dots, t_n, y_n) \quad \text{if } t_i = 0, \\ &= M_{i-1}(y_0, t_1, y_1, \dots, t_{i-1}, y_{i-1})M_{n-i}(y_i, t_{i+1}, \dots, t_n, y_n) \quad \text{if } t_i = 1. \end{aligned}$$

We specialize this definition to call f *m-homotopy-multiplicative* if the M_n exist for $0 \leqq n \leqq m$. If X is a topological monoid, X will be called *strongly homotopy-commutative in the sense of Sugawara*, if the multiplication $m\colon X \times X \to X$ is a strongly homotopy-multiplicative map.

EXAMPLE 4. If X is a monoid which is strongly homotopy-commutative in the sense of Sugawara, then X is a C_∞-space. It was pointed out to me by J. D. Stasheff that this fact follows from triangulating the K_n's and setting sufficiently many coordinates equal to the basepoint in Sugawara's definition to obtain maps $\Delta_n \times X^n \to X$, then piecing these together. Since this example is essentially equivalent to Example 3, no further details will be given concerning it.

The category of countable CW-complexes which are monoids, and n-homotopy-multiplicative maps is the true domain of the C_n-spaces, in the sense indicated by the following two results, whose proofs will be given in §3.

PROPOSITION 8. *Let Y and Z be monoids, where Y is countable CW-complex, and Z is a C_n-space. Let $f\colon Y \to Z$ be an n-homotopy-multiplicative map which is a weak homotopy equivalence (i.e., $f_*\colon \pi_*(Y) \to \pi_*(Z)$ is an isomorphism). Then Y may be given a C_n-form.*

From this proposition follows immediately the following invariance theorem.

THEOREM 9. *In the category of countable CW-monoids, the property of being a C_n-space is an invariant of n-homotopy-multiplicative homotopy type.*

REMARK 10. Let (K, L) be a pair of complexes. Then L is said to be *retractile* in K, according to [5], provided N is contractible in M, where (M, N) is the identification space $(CK, K)/(L, L)$. The following facts about retractile subspaces will be applied to the C_n-theory.

(1) Let X be an H-space. If L is retractile in K, then given $f\colon CY \times K \to X$ and $h\colon CY \times L \to X$ such that $f \mid Y \times L = h \mid Y \times L$, then $f \mid Y \times K$ extends to $f\colon CY \times K \to X$ such that $f \mid CY \times L = h$.

(2) The subspace $X^{[i]}$ is retractile in X^i.

It follows from these facts that if $\{Q_i\}$, $i=1,\dots,n-1$, is a C_{n-1}-form on a countable CW-monoid, X, and if $Q'_n\colon K_n\times X^n\to X$ satisfies condition (2) of the definition of C_n-forms, then there exists $Q_n\colon K_n\times X^n\to X$ such that $\{Q_i\}$, $i=1,\dots,n$, is a C_n-form on X.

The main results on C_n-spaces relate the algebraic property given by the C_n-form to certain geometric properties of X. These results may be phrased in terms of quasifibrations. Hence, recall the following definition.

DEFINITION 11. A continuous surjection $p\colon E\to B$ is called a *quasifibration*, q.f. for short, provided that

$$p_*\colon \pi_*(E, p^{-1}(x), y)\to \pi_*(B, x)$$

is an isomorphism for all x in B and y in $p^{-1}(x)$.

DEFINITION 12. A quasifibration $p\colon E\to B$ is called *principal*, provided that the fiber $X=p^{-1}(*)$ is a monoid which acts on E in such a manner that (if $N\colon X\times E\to E$ denotes the action):

(1) $N(*, y)=y$,

(2) $N(X\times\{y\})\subset p^{-1}(p(y))$,

(3) $N(m(x, x'), y)=N(x, N(x', y))$, and

(4) $N(x, x')=m(x, x')$,

for all x, x' in X and all y in Y. Here m denotes the multiplication of X.

DEFINITION 13. Let X be a space. Then $(X)_n$, *the* (James) *n-fold reduced* product space of X, is obtained by identifying points of X^n with each other if and only if they are the same when occurrences of the base point are disregarded. There is an obvious inclusion $(X)_n\subset(X)_{n+1}$ and the space obtained by taking the union of all $(X)_n$ under these inclusions (with the weak topology) is called $(X)_\infty$. (It is well known that $(X)_\infty$ is the homotopy type of ΩSX, cf. [6], whenever X is a connected CW-complex.)

MAIN THEOREM 14. *Let X be a countable CW-monoid. Then the following conditions are equivalent.*

(1) *X admits a C_n-form.*

(2) *The Hopf construction for X, $p_2\colon E_2\to SX$, extends to a principal quasifibration $p\colon E_n\to B_n$, where B_n is the homotopy type of $(SX)_n$.*

(3) *There is an n-homotopy-multiplicative map $d\colon \Omega(SX)_n\to X$ such if $j\colon X\to \Omega(SX)_n$ is the usual inclusion given by $j(x)[t]=t\wedge x$ in SX, considered as a subspace of $(SX)_n$, then dj is homotopic to the identity map of X.*

§3 will be devoted to the proof of this theorem, but we present here an outline of the proof. The implication (1) $\Rightarrow$ (2) will be proven by means of a direct construction in the Dold-Lashof vein.

The implication (2) $\Rightarrow$ (3) will be derived from the next two results.

LEMMA 15. *If $p: E \to B$ is any q.f., there is a "connecting map" $d: \Omega B \to F$, where F is the fiber. If p extends the Hopf map p_2, then dj is homotopic to the identity map of F.*

THEOREM 16. *If $p: E \to B$ is any principal q.f., then the connecting map $d: \Omega B \to F$ is strongly homotopy-multiplicative.*

Lemma 15 is standard, and the proof of Theorem 16 may be found in [3].

There is a well known concept of what is meant by a sub-H-space of an H-space being homotopy-commutative in the containing space, see [7]. In §3 there will be given an analogous definition of a submonoid, Y, being C_n in a containing monoid, X. The implication (3) ⇒ (1) is based on the following lemma.

LEMMA 17. *Let the inclusion $\Omega SX \subset \Omega(SX)_n$ be induced by the inclusion $SX \subset (SX)_n$. Then ΩSX is C_n in $\Omega(SX)_n$.*

An application of our Main Theorem is the following corollary.

COROLLARY 18. *A countable CW-complex, X, is a C_∞-space if and only if there is a strongly homotopy-multiplicative map $d: \Omega^2 S^2 X \to X$ such that dj is homotopic to the identity, where $j: X \to \Omega^2 S^2 X$ is given by the composition $X \to \Omega SX \to \Omega^2 S^2 X$.*

REMARK 19. There is another characterization of C_n-spaces which, owing to the length of its proof, will merely be noted here, its development being reserved for a subsequent paper. It is as follows: *A countable CW-monoid, X, is a C_n-space if and only if there is a map $a: (SX)_n \to P_n(X)$ which extends the identity map $1: SX \to SX = P_1(X)$.* Here $P_n(X)$ is the nth projective space of X, as constructed in [1].

This characterization is significant in the following two respects. First, it extends the theorem of Stasheff, from [11], that a homotopy-associative H-space is homotopy-commutative if and only if the map $a: (SX)_2 \to P_2(X)$ exists. Secondly, such a map $a: (S^2)_n \to P_n(S^1)$, obtained by other means, is exploited by Toda, p. 174 of [16], in his computations of homotopy groups of spheres. Thus the existence of his map seems to be related to the commutativity of S^1.

The idea of the proof of this characterization is to obtain a fiber-preserving map from the total space, E_n, of Theorem 14, to the total space of the Dold-Lashof construction, E_n.

It is a natural question to ask what the maps are in the category of C_n-spaces, i.e., what are the maps which sufficiently respect the C_n-forms. Just as in the category of H-spaces it is more useful to consider the H-maps rather than only the strict homomorphisms, the maps in the C_n-category will be taken to be those which preserve the structure up to homotopy.

In his work with A_n-spaces, Stasheff in [12] determines necessary and sufficient conditions to decide when the total space of a fibration induced from a path space fibration admits an A_n-form. His methods serve to motivate the following definitions and to provide models for the resulting theorems, which answer the analogous questions for C_n-forms.

DEFINITION 20. Let $(X, \{Q_i^X\})$ and $(W, \{Q_i^W\})$ be C_n-spaces. Then a map $f: X \to W$ is called a *C_n-map*, provided that f is a homomorphism of monoids and that there exist maps

$$D_i: I \times K_i \times X^i \to W, \qquad i = 2, \ldots, n,$$

such that

$$(1) \qquad \begin{aligned} D_i(0, \tau, x_1, \ldots, x_i) &= Q_i^W(\tau, f(x_1), \ldots, f(x_i)), \\ D_i(1, \tau, x_1, \ldots, x_i) &= f(Q_i^X(\tau, x_1, \ldots, x_i)); \end{aligned}$$

$$(2) \qquad \begin{aligned} &D_i(t, V(A_r, B_s)[\rho, \sigma], x_1, \ldots, x_i) \\ &\qquad = D_r(t, \rho, x_{A(1)}, \ldots, x_{A(r)}) \cdot D_s(t, \sigma, x_{B(1)}, \ldots, x_{B(s)}) \end{aligned}$$

and

$$(3) \qquad D_i(t, \tau, x_1, \ldots, x_i) = D_{i-1}(t, s_j(\tau), x_1, \ldots, \hat{x}_j, \ldots, x_i)$$

if $x_j = e$, the identity of X.

If X and W are C_∞-spaces, and the D_i exist for all $i \geq 1$, then f is called a *C_∞-map*.

EXAMPLES.

(1) A homomorphism of abelian monoids is a C_∞-map.

(2) Let X and Y be H-spaces and $f: X \to Y$ be an H-map. Then $f: \Omega X \to \Omega Y$ is a *C_∞-map*.

The following theorem illustrates the use of C_n-maps by giving an extension theorem for C_n-forms.

THEOREM 21. *Let X and W be countable CW C_n-spaces and $f: X \to W$ be a C_n-map. Let $p: Y \to X$ be the fibration induced by f from the standard fibration $\pi: PW \to W$. Then Y may be given the structure of a C_n-space in such a way that p is a C_n-map.*

Partial converses to this theorem may be easily obtained by making dimensionality restrictions on the homotopy and cohomology groups of the spaces involved in order to insure that certain obstructions vanish. The next two theorems, which are of a rather technical nature, illustrate this method.

THEOREM 22. *Let $p: Y \to X$ be the fibration induced from $\pi: PW \to W$ by a map $f: X \to W$. Suppose that X, Y, and W are C_n-spaces, that p is a C_n-map, and that f is a C_{n-1}-map. Further, suppose that there exist integers p and q, $q \geq p \geq 2$ such that*

(1) $\pi_i(X) = 0$, for $i \leq p-1$, and

(2) $\pi_i(W) = 0$, for $i \leq q$ and $i \geq (n-1)p+q$.

Then f is a C_n-map.

THEOREM 23. *Let $(X, \{Q_i^X\})$ be a C_{n-1}-space and let $(Y, \{Q_i^Y\})$ be a C_n-space. Let $p: Y \to X$ be a fibration with fiber F which is a C_{n-1}-map. Further, suppose that there exist integers p and q, $q \geq p \geq 2$, such that*

(1) $\pi_i(X) = 0$, for $i \leq p-1$ and $i \geq (n-1)(p+1)+q$, and

(2) $\pi_i(F) = 0$, for $i \leq q-1$.

Then there exist $Q_n^K: K_n \times X^n \to X$ *such that*

(a) $\{Q_i^X\}$ *is a* C_n*-form on* X, *and*

(b) p *is a* C_n*-map.*

The proofs of the last three theorems will be outlined in §3. These theorems, together with Proposition 8 yield the following application, cf. [12, pp. 301–302].

THEOREM 24. *A space with a two-stage Postnikov system admits a* C_n*-form if and only if its k-invariant is represented by a* C_n*-map.*

3. Let us begin this section by considering Example 3 of the preceding section, namely that the space of loops on an H-space is a C_∞-space. Its proof will follow from the proof of Lemma 17. We need the following definition.

DEFINITION 25. Let Y be a monoid, X a space. A map $\psi: Z \to Y$ is called a C_n*-commutativity of* Z *into* Y if there are maps

$$Q_i: K_i \times Z^i \to Y, \qquad i = 1, \dots, n$$

satisfying conditions (2) and (3) of the definition of C_n-form, but with condition (1) replaced by

(1) $Q_1: Z \to Y$ is just $\psi: Z \to Y$.

We now restate Lemma 17.

LEMMA 17. *Let* $\psi: \Omega X \to \Omega(X)_n$ *be induced by the inclusion* $X \to (X)_n$. *Then* ψ *is a* C_n*-commutativity of* ΩX *into* $\Omega(X)_n$.

Proof. This proof is an extension of the familiar one showing that the loop-space of an H-space is homotopy-commutative. It is based on the fact that there are the two "multiplications" of ΩX in $\Omega(X)_n$. One is the loop additions, and the other is given by $(\lambda_1^{r_1}, \lambda_2^{r_1}) \to [\lambda_1^{r_1}, \lambda_2^{r_2}]$, where $[\lambda_1^{r_1}, \lambda_2^{r_2}]$ is the loop in $(X)_n$ of length $\max(r_1, r_2)$

$$\begin{aligned}[\lambda_1^{r_1}, \lambda_2^{r_2}](t) &= [\lambda_1^{r_1}(t), \lambda_2^{r_2}(t)], \quad t = \min(r_1, r_2),\\ &= [\lambda_1^{r_1}(r_1), \lambda_2^{r_2}(t)], \quad \text{if } r_1 = \min(r_1, r_2),\\ &= [\lambda_1^{r_1}(t), \lambda_2^{r_2}(r_2)], \quad \text{if } r_2 = \min(r_1, r_2).\end{aligned}$$

Furthermore, $[\lambda_1^{r_1}, \dots, \lambda_i^{r_i}]$ is defined for $i \leq n$ by iteration. Now assume inductively that Q_j has been defined for $j < i$ in such a manner that

$$Q_i(\tau, \lambda_1^{r_1}, \dots, \lambda_i^{r_i}) = [\dots, e^{a_j} + \lambda_j^{b_j} + e^{c_j}, \dots],$$

when given by requirement (2) of the definition. Here $\lambda_j^{b_j}$ denotes $\lambda_j^{r_j}$ reparametrized to the length b_j and e^{a_i} is the constant path at the basepoint of length a_i.

Then we may regard K_i as the cone on its boundary and set

$$Q_i(\tau \wedge s, \lambda_1^{r_1}, \dots, \lambda_i^{r_i}) = [\dots, e^{sa_j} + \lambda_j^{b_j + (1-s)(a_j + c_j)} + e^{sc_j}, \dots],$$

which accomplishes the definition of Q_i.

As a corollary, we have Example 3.

COROLLARY 26. *If X is an H-space, then ΩX is a C_n-space for all n.*

Proof. Let $(X)_n \xrightarrow{\Phi} X$ be given by multiplication, associating from the right. Then the composition

$$K_i \times (\Omega X)^i \xrightarrow{Q_i} \Omega(X)_n \xrightarrow{\Omega\phi} \Omega X, \qquad 1 \leqq i \leqq n$$

is a C_n-form in X, where Q_i is the map of the preceding lemma.

We now turn to the proof of Proposition 8. It will be accomplished by induction on i, the degree of the C_i form, then by induction on the skeleta of X.

Proof of Proposition 8. Suppose $Q_j^X\colon K_j \times X^j \to X$ has been defined for $j<i$, in such a way that there is a homotopy, H_j, between $f \circ Q_j^X$ and $Q_j^Y \circ (1 \times f^j)$ satisfying the consistency hypothesis stated below. Our problem is to construct a C_i-form on X. Its value is already specified on $Q_i^X\colon \mathrm{Bd}\,(K_i) \times X^i \to X$.

It suffices to show that there is a homotopy $H_i\colon I \times \mathrm{Bd}\,(K_i) \times X^i \to Y$ between $f \circ Q_i^X$ and $Q_i^Y \circ (1 \times f^i)$. Then making use of the fact that f induces isomorphisms of homotopy groups, Q_i^X may be extended by induction on the skeleta of $K_i \times X^i$ in such a way that H_i extends to become a homotopy between $f \circ Q_i^X$ and $Q_i^Y \circ (1 \times f^i)$ on all of $I \times K_i \times X^i$.

Such a homotopy exists on each top-dimensional face of $\mathrm{Bd}\,(K_i)$, but these homotopies may not coincide on the intersections of the faces. Specifically suppose $\sum_{j=1}^k r_j = i$ and faces of the form $K_{r_1+\cdots+r_j} \times K_{r_{j+1}+\cdots+r_k}$ intersect; $j=1,\ldots,k-1$. We may write their intersection as $K_{r_1} \times \cdots \times K_{r_k}$. Let $(\rho_1,\ldots,\rho_k)$ be a point in this intersection. Then for each j, we have a homotopy

$$\begin{aligned}
f(Q_{r_1}^X(\rho_1, x_{A^1(1)}, \ldots, x_{A^1(r_1)}) \cdot \cdots \cdot Q_{r_k}^X(\rho_k, x_{A^k(1)}, \ldots))& \\
= f(Q_\alpha^X(\rho_\alpha, x_{A^\alpha(1)}, \ldots, x_{A^\alpha(\alpha)}) \cdot Q_\beta^X(\rho_\beta, x_{A^\beta(1)}, \ldots))& \\
\sim f(Q_\alpha^X(\rho_\alpha, x_{A^\alpha(1)}, \ldots) \cdot f(Q_\beta^X(\rho_\beta, x_{A^\beta(1)}, \ldots))& \\
\sim Q_\alpha^Y(\rho_\alpha, f(x_{A^\alpha(1)}), \ldots) \cdot Q_\beta^Y(\rho_\beta, f(x_{A^\beta(1)}), \ldots),& \text{ by } D_\alpha \times D_\beta, \\
= Q_{r_1}^Y(\rho_1, f(x_{A^1(1)}), \ldots) \cdot \cdots \cdot Q_{r_k}^Y(\rho_k, f(x_{A^k(1)}), \ldots).&
\end{aligned}$$

Here $\alpha = r_1 + \cdots + r_j$, $\beta = r_{j+1} + \cdots + r_k$, and (A^α, A^β) is the (α,β) partition which refines to the $(r_1,\ldots,r_k)$ partition $(A^1,\ldots,A^k)$.

Then in order to piece together the homotopies, exactly what is required is that f be a k-homotopy-multiplicative map.

Now let us prove Theorem 13.

Proof that (1) ⇒ (2). We begin with $E_0 = X$ and $a_1\colon X \to X$ the identity map. Let $Z_i = (K_i \times X \times X^{[i-1]}) \cup (\mathrm{Bd}\,(K_i) \times X^i) \subset K_i \times X^i$. Here $X^{[i-1]}$ denotes the subspace of X^{i-1} consisting of those points at least one of whose coordinates is the basepoint. Then $p_i\colon E_i \to B_i$, $i \leqq n$, are constructed inductively as follows: Let

$$a_i\colon (K_i \times X^i, Z_i) \to (E_{i-1}, E_{i-2})$$

be a relative homeomorphism, in which $a_i | Z_i$ is given by

(1) if $x_j = e$, the identity of X, then

$$a_i(\tau, x, x_1, \ldots, x_{i-1}) = a_{i-1}(s_j(\tau), x, x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}),$$

(2) if (A_r, B_s) is a partition of i, and $i \in A_r$, then

$$a_i(V(a_r, B_s)[\rho, \sigma], x, x_1, \ldots, x_{i-1}) = a_r(\rho, x, x_{A'(1)}, \ldots, x_{A'(r-1)}),$$

where $A'_{r-1} = A_r - (i)$, and

(3) if (A_r, B_s) is a partition of i, and $i \in B_s$, then

$$a_i(V(A_r, B_s)[\rho, \sigma], x, x_1, \ldots, x_{i-1}) \\ = a_s(\sigma, x \cdot Q_r(\rho, x_{A(1)}, \ldots, x_{A(r)}), x_{B'(1)}, \ldots, x_{B'(s-1)})$$

where $B'_{s-1} = B_s - (i)$.

Let B_0 be a point. Then let $b_i\colon (K_i \times X^{i-1}, W_i) \to (B_{i-1}, B_{i-2})$ be a relative homeomorphism, where

$$W_i = (K_i \times X^{[i-1]}) \cup (\mathrm{Bd}\,(K_i) \times X^{i-1})$$

and $b_i | W_i$ is defined by formulas (1), (2), and (3) with x omitted. Now define $p_i\colon E_i \to B_i$ as being induced from the projection $K_i \times X \times X^{i-1} \to K_i \times (e) \times X^{i-1} = K_i \times X^{i-1}$.

The action $N_i\colon X \times E_i \to E_i$ is defined by

$$N_i(z, \alpha_i(\tau, x, x_i, \ldots, x_{i-1})) = \alpha_i(\tau, zx, x_1, \ldots, x_{i-1}).$$

First we verify that B_n is homotopic to $(SX)_n$. Recall that $(SX)_n$ may be formed by relative homeomorphisms:

$$c_i\colon (I^i \times X^i, (\mathrm{Bd}\,(I^i) \times X^i) \cup (I^i \times X^{[i]})) \to ((SX)_i, (SX)_{i-1}), \qquad i = 1, \ldots, n,$$

given by

$$c_i(t_1, \ldots, t_i, x_1, \ldots, x_i) = c_{i-1}(t_1, \ldots, \hat{t}_k, \ldots, t_i, x_1, \ldots, \hat{x}_k, \ldots, x_i),$$
$$\text{if } t_i = 0 \text{ or } 1, \text{ or if } x_k = e.$$

Now consider the map $g_i\colon K_i \to I^{i-1}$ given on the vertices of K_i as follows. Let $(r_1, \ldots, r_i)$ be a vertex of K_i given by an element $\gamma \in S(i)$ by $(r_1, \ldots, r_i) = \gamma(1, \ldots, i)$. Then set $g_i(r_1, \ldots, r_i) = (t_2, \ldots, t_i) \in I^{i-1}$ where

$$\begin{aligned} t_k &= 0, \quad \text{if } \gamma(1) < \gamma(i), \\ &= 1, \quad \text{if } \gamma(i) < \gamma(1). \end{aligned}$$

It is easily verified that the image of the set of vertices of a face of K_i is the set of vertices of a face of I^{i-1}. Hence g_i can be extended by linearity to map $\mathrm{Bd}\,(K_i)$ onto $\mathrm{Bd}\,(I^{i-1})$. Furthermore, it is clear how to deform g_i into a homeomorphism. Now extend g_i to map $g_i\colon K_i \to I^i$ in such a way that g_i restricted to $\mathrm{Int}\,(K_i)$ is a homeomorphism. Let $\psi_1\colon B_1 \to SX$ be the identity map. We shall define inductively maps $\psi_i: B_i \to (SX)_i$ such that $\psi_i | B_{i-1} = \psi_{i-1}$. Suppose we have done this for $j < i$ in such a way that the diagram

$$\begin{array}{ccc} W_{i+1} & \xrightarrow{g_{i+1}\times 1} & (\mathrm{Bd}\,(I^i)\times X^i) \cup (I^i \times X^{[i]}) \subset I^i \times X^i \\ \downarrow b_{i+1} & & \downarrow c_i \\ B_{i-1} & \xrightarrow{\psi_{i-1}} & (SX)_{i-1} \end{array}$$

is commutative and such that ψ_{i-1} is a homotopy equivalence. Then ψ_{i-1} can be extended to a homotopy equivalence, $\psi_i: B_i \to (SX)_i$. The fact that this ψ_i satisfies the inductive hypothesis follows from the definitions of g_{i+2}, b_{i+2}, and c_{i+1}.

To complete the implication (1) $\Rightarrow$ (2), it remains to show that $p_i: E_i \to B_i$ is a quasifibration. The method of proof is due to Dold and Thom [2]. The base, B_i, may be expressed as the union of two open subsets, U and W, such that p_i is a q.f. over each of U, W, and $U \cap W$. The details are similar to those in the proof of Theorem 19 on p. 286 of [12] and hence will be omitted.

As mentioned in §2, the implication (2) $\Rightarrow$ (3) follows immediately from Lemma 15 and Theorem 16. We now consider the implication (3) $\Rightarrow$ (1).

Proof that (3) $\Rightarrow$ (1). By Lemma 17, there are maps $\hat{Q}_i: K_i \times (\Omega SX)^i \to \Omega(SX)_n$, $i = 1, \ldots, n$, realizing the inclusion $\Omega SX \subset \Omega(SX)_i$ as a C_n-commutativity of ΩSX into $\Omega(SX)_i$. Composition with the canonical inclusion of X into ΩSX yields a C_n-commutativity of X into $\Omega(SX)_n$ realized by maps $\bar{Q}_i: K_i \times X^i \to \Omega(SX)_i$. Our first attempt to define a C_n-form on X would be to define $Q_i: K_i \times X^i \to X$ to be the composition

$$K_i \times X^i \xrightarrow{\bar{Q}_i} \Omega(SX)_i \xrightarrow{d} X.$$

This attempt would succeed if d were actually a homomorphism. Since d is only an n-homotopy-multiplicative map, however, this definition does not work. Instead, the Q_i have to be defined inductively, using a procedure similar to that of the proof of Proposition 9. (An alternative approach would be to replace d by an equivalent map which *is* a homomorphism.) With this modification, the construction of the C_n-form on X can be completed. This finishes the proof of Theorem 14.

Now we turn to the results on C_n-maps and on C_n-forms in induced fibrations. We begin by remarking that the claim in Example 2 that the loop map of an H-map is a C_∞-map is easily verified using the same homotopies which were used in constructing the C_∞-forms on ΩX and ΩY. We now proceed to the next theorem.

Proof of Theorem 21. Let m^X, Q_i^X, m^W, Q_i^W, be the multiplications and C_i-forms on X and W, respectively. We are given maps

$$D_i: I \times K_i \times X^i \to W, \qquad 2 \leq i \leq n.$$

Let

$$\hat{D}_i: K_i \times X^i \to W^I \times R \qquad (R \text{ the reals})$$

be the adjoint of D_i, deformed to make it basepoint-preserving, keeping the deformation through paths satisfying the boundary conditions of Definition 21.

We may use the multiplication and C_i-forms on W to induce laws of composition, $\hat{m}^W$ and $\hat{Q}^W$, on W^I, in similar fashion to that in the proof of Lemma 17.

The multiplication m^Y, on Y and the C_i-forms, Q_i^Y, on Y may be given as follows:

$$m^Y((x_1, a_1), (x_2, a_2)) = (m^X(x_1, x_2), \hat{m}^W(a_1, a_2)),$$

and

$$\begin{aligned} &Q_i^Y(\tau, (x_1, a_1), \ldots, (x_i, a_i)) \\ &\quad = (Q_i^X(\tau, x_1, \ldots, x_i), \hat{Q}_W^i(\tau, a_1, \ldots, a_i) + \hat{D}_i(\tau, x_1, \ldots, x_i)). \end{aligned}$$

Here $\tau \in K_i$, $x_i, \ldots, x_i \in X$, and $a_1, \ldots, a_i \in PW$.

The verification that these definitions satisfy the requirements is straightforward.

We now turn to the proofs of Theorems 22 and 23.

Proof of Theorem 22. What is required is a map $D_n: I \times K_n \times X^n \to W$ which satisfies the boundary conditions as given in Definition 20. These conditions already determine D_n on the subset Bd $(I \times K_n) \times X^n \cup I \times K_n \times X^{[n]}$.

The proof that the obstructions to extending D_n to the rest of $I \times K_n \times X^n$ are zero uses the same methods as those in Proposition 10.5 on p. 309 of [12] and thus will be omitted. The proof of Theorem 23 uses the same techniques as Theorem 22 in locating the obstructions to the construction of Q_n in trivial cohomology groups, and consequently we again refer to Stasheff's work in [12] for the methods of this proof.

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